The Harmonic oscillator

You may be familiar with several examples of harmonic oscillators form classical mechanics, such as particles on a spring or the pendulum for small deviation from equilibrium, etc.

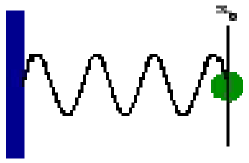

Figure 4.1: The mass on the spring and its equilibrium position

Let me look at the characteristics of one such example, a particle of mass m on a spring. When the particle moves a distance x away from the equilibrium position x_0, there will be a restoring force $-kx$ pushing the particle back ($x > 0$ right of equilibrium, and $x < 0$ on the left). This can be derived from a potential

$$V(x) = \frac{1}{2}kx^2 \tag{4.1}$$

Actually we shall write $k = m\omega^2$. The equation of motion

$$m\ddot{x} = -m\omega^2 x \tag{4.2}$$

has the solution

$$x(t) = A\cos(\omega t) + B\sin(\omega t) \tag{4.3}$$

We now consider how this system behaves quantum-mechanically.

Wave packets

The direct connection between the physical interpretation of $\phi_k(x)$ as a beam of particles and our formal theory comes through the TDSE. The pure states $\phi_k(x)$ of energy are stationary, and the connection between them and the dynamical evolution pictured is not direct. Producing a wave packet with an identifiable location in space requires taking superpositions of the states $\phi_k(x)$.

Mathematically, a wave packet evolving under the TDSE is such a superposition of pure energy states times multiplied with the appropriate time-dependent phase factors,

$$\Psi(x,t) = \int \frac{dk}{\sqrt{2\pi}}\, \tilde{\psi}(k) e^{ikx} e^{-i\frac{\hbar k^2}{2m}t}, \tag{4}$$

where $|\psi(k)|$ is sharply peaked near $k = k_0$. The mathematical form of the integral (4) naturally guarantees that the resulting wave packet $\Psi(x,t)$ will be confined to a particular region of space because the integral is essentially essentially a sum of complex numbers with varying phases.

For most values of x, these phases vary rapidly, resulting in much cancellation and a small absolute value of the integral.

To sketch the behavior of the integral, we write $\psi(k)$ as the product of its amplitude and a complex phase

$$\tilde{\psi}(k) \equiv |\tilde{\psi}(k)| e^{i\phi_o(k)},$$

where $\phi_o(k)$ describes the phase of the packet. Note $|\psi(k)|$ is peaked about $k = k_o$ as in Figure 2.

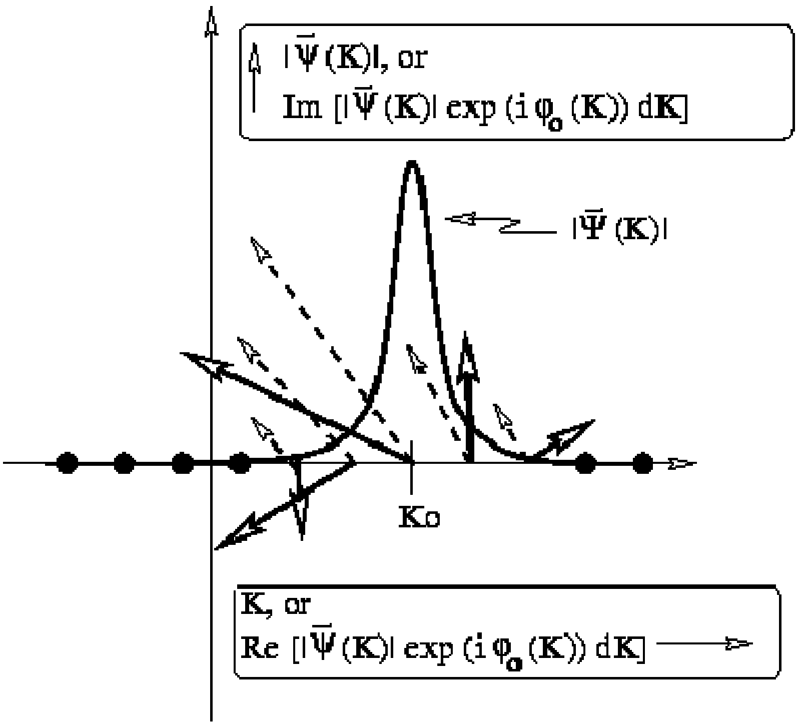

Figure 2: Integrand of an integral to be analyzed using the method of stationary phase

With this separation we may rewrite (4) as the integral of the product of real amplitudes with complex phases,

$$\Psi(x,t) = \int dk \, \frac{|\tilde{\phi}(k)|}{\sqrt{2\pi}} e^{i\phi_o(k)} e^{ikx} e^{-i\frac{\hbar k^2}{2m}t}. \qquad (5)$$

Such integrals are best analyzed using the method of stationary phase as described in the next section.